Fisher, bound, and extreme physical information for dissipative processes

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(Received 21 January 2003; published 25 July 2003)

In the present paper we discuss the possible form and meaning of Fisher, bound, and physical information in some special cases. It seems to us that an unusual choice of bound information may describe the behavior of dissipative processes.

DOI: 10.1103/PhysRevE.68.016121 PACS number(s): 05.70.Ln, 89.70.+c, 04.20.Fy, 11.10.Ef

I. INTRODUCTION

Information has been found to play an increasingly important role in physics, mainly since Jaynes' pioneering work [1], which is a discussion of connection of information theory and statistical mechanics. Knowledge of probability p—representing the observer's state of knowledge about the system, rather than the state of the system itself—enables us to express the so-called Fisher information [2–7]. This is a quality of an efficient measurement procedure, and it is also a measure of the degree of system disorder, in other words, it is a form of entropy,

$$I=\int \frac{(\nabla p)^2}{p} d^3x. \tag{1}$$

Here, p denotes the probability density function for the noise value x, and ∇ is the gradient operator. Fisher found that it is often more convenient to calculate with a real amplitude function $q(x,t)$ [4], where

$$p=\frac{1}{8}q^2, \tag{2}$$

by which we can write the Fisher information

$$I=\frac{1}{2}\int (\nabla q)^2 d^3x. \tag{3}$$

Any measurement of physical parameters initiates a transformation of Fisher information. An information transition $J \to I$ takes place, where J represents the physical effect. J is the information that is intrinsic to the phenomenon. In general, the information J is identified by an invariance that characterizes the measured phenomenon. As a basic case, we consider

$$J=\frac{1}{2}\int \dot{q}^2 d^3x, \tag{4}$$

where the dot denotes the time derivative. The possibility of some loss of information during the information transition suggests

$$I\leqslant J. \tag{5}$$

The principle of extreme physical information represents a kind of game between the observer and nature. The observer wants to maximize I while nature wants to minimize it. The physical information K is

$$\begin{aligned} K&=I-J\\ &=\frac{1}{2}\int (\nabla q)^2 dx-\frac{1}{2}\int \dot{q}^2 d^3x\\ &=\int \left(\frac{1}{2}(\nabla q)^2-\frac{1}{2}\dot{q}^2\right) d^3x, \end{aligned} \tag{6}$$

which is a loss of information. Considering the time evolution of the process, it has an extremum, which formulates a variational principle for finding the q. From this equation we can read the so-called Lagrange density funtion [8],

$$L=\frac{1}{2}(\nabla q)^2-\frac{1}{2}\dot{q}^2, \tag{7}$$

which is the integrand of the above equation. The equation of motion can be deduced from Hamilton's principle (the action $S=\int L d^3x dt=\text{extremum}$) by the help of calculus of variations (i.e., $\delta S=0$). From the extremization of functional S, we can obtain a partial differential equation as Euler-Lagrange equation,

$$\ddot{q}-\Delta q=0, \tag{8}$$

which is the well-known wave equation. (Δ denotes the Laplace operator.)

We restrict our attention and examination to the above described basic case. Detailed examination of the principle of extreme physical information and several examples can be found in Refs. [2–7]. Special examples, applications, and advanced results from the standpoint of information can be found for Higgs mass generation [9], physical properties of a generally decoherent system [10], a Schrödinger link between nonequilibrium thermodynamics and Fisher informa-

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tion [11], nonequilibrium thermodynamics and Fisher information [12], and a great number of other papers.

II. ANOTHER CHOICE OF PROBABILITY

We can find the probability p in a quadratic form like in Eq. (2), but we give it by a different inner function. Now, let the probability be proportional to the square of gradient of a function $\varphi(x,t)$,

$$p=\frac{1}{8}(\nabla\varphi)^2, \tag{9}$$

where φ is a generalized potential function. (We point out an interesting application of this function for the case of dissipative processes, e.g., heat conduction. This potential function was originally introduced to apply the Hamilton's principle and Lagrange formalism for irreversible processes [13].) Using Eqs. (1) and (9) the Fisher information can be given by

$$I=\frac{1}{2}\int(\Delta\varphi)^2 d^3x. \tag{10}$$

Similar to the previous example, we write the bound information J in the form

$$J=\frac{1}{2}\lambda\int\dot{\varphi}^2 d^3x, \tag{11}$$

where λ is a constant parameter. The extreme physical information K can be formulated,

$$K=I-J=\int\left(\frac{1}{2}(\Delta\varphi)^2-\frac{1}{2}\lambda\dot{\varphi}^2\right)d^3x, \tag{12}$$

from which the Lagrange density function is obtained,

$$L=\frac{1}{2}(\Delta\varphi)^2-\frac{1}{2}\lambda\dot{\varphi}^2. \tag{13}$$

After the variation, the equation of motion can be calculated as Euler-Lagrange equation

$$\lambda\ddot{\varphi}+\Delta\Delta\varphi=0. \tag{14}$$

What may λ be? If $\lambda=1$ (the choice of other positive number gives physically the same result),

$$J=\frac{1}{2}\int\dot{\varphi}^2 d^3x. \tag{15}$$

In this case the Lagrange density function is

$$L=\frac{1}{2}(\Delta\varphi)^2-\frac{1}{2}\dot{\varphi}^2, \tag{16}$$

by which the field equation can be calculated,

$$\ddot{\varphi}+\Delta\Delta\varphi=0. \tag{17}$$

This equation is valid for free oscillations of a thin plate or rod [14]. These waves are fundamentally different from those in a medium in all directions. Considering a monochromatic elastic wave ($\varphi\sim\exp[i(kx-\omega t)]$), we obtain the dispersion relation in the form $\omega=k^2$. The propagating wave is not dissipative in spite of the special behavior of dispersion relation. If $\lambda=-1$, then the bound information is always negative,

$$J=-\frac{1}{2}\dot{\varphi}^2. \tag{18}$$

Here, we can see that the condition $I\leqslant J$ [see Eq. (5)] is immediately violated. It is not clear what it means at all, but we can examine the mathematical results that follow from this assumption. The Lagrange density function can be written as

$$L=\frac{1}{2}(\Delta\varphi)^2+\frac{1}{2}\dot{\varphi}^2, \tag{19}$$

which is the basic funtion of diffusive processes (e.g., linear heat conduction) in the field theory of nonequilibrium thermodynamics [15,16]. This proves that there exists such a physical system where this choice of J is relevant. We obtain a biparabolic differential equation as Euler-Lagrange equation,

$$\ddot{\varphi}-\Delta\Delta\varphi=0. \tag{20}$$

As it has been shown (in Refs. [15,16]), a new quantity $T(x,t)$ can be introduced (this is the local equilibrium temperature, but it may be the concentration, etc.),

$$T=-\dot{\varphi}\Delta\varphi. \tag{21}$$

Equations (20) and (21) are equivalent to

$$\dot{T}-\Delta T=0, \tag{22}$$

which is the Fourier equation. The diffusive processes are dissipative, the observed system tends to the static state. Is this the meaning of the negative bound information or is it a fortunate accident? We have not known it yet. Here, one can ask whether it was possible to use Eqs. (15)–(17) (avoiding the assumption of negative J) to obtain the equation of heat conduction by a different substitution. It is easy to see that none of the combinations, i.e., $T=-\dot{\varphi}+\Delta\varphi$ or $T=\dot{\varphi}-\Delta\varphi$, can give Eq. (22). To understand the meaning of the different possibilities of choice of sign of J, we turn back to the basic problem, but we write

$$J=-\frac{1}{2}\int\dot{q}^2 d^3x. \tag{23}$$

The Lagrange density function can be obtained as

$$L=\frac{1}{2}(\nabla q)^2+\frac{1}{2}\dot{q}^2, \tag{24}$$

by which we calculate the equation of motion (an elliptic differential equation)

$$\ddot{q}+\Delta q=0. \tag{25}$$

The solution of this equation can be calculated

$$q=Ae^{-\beta x}e^{i\omega t}+Be^{-\kappa t}e^{ikx}, \tag{26}$$

which includes the dissipation in the second term. (A, B, β, ω, κ, and k are constant parameters.) It seems to us, similar to the heat conduction, that the negative bound information J may have a connection with the dissipation.

III. CONCLUSION

In the case of dissipative processes the bound information J is negative. This is a rather strange thing for the first view, however, it may mean that all physical information will be lost in the process. This comes from the definition of K; moreover, this may show the connection of dissipation and information loss. During the process the system tends to the equilibrium state, which means that more information cannot be obtained about the phenomenon.

ACKNOWLEDGMENT

Katalin Gambár would like to thank the Bolyai János Research Foundation of the Hungarian Academy of Sciences for financial support.

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